

Entanglement generation in spatially separated systems using quantum walk

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We present a novel scheme to generate entanglement between two spatially separated systems. The scheme makes use of spatial entanglement generated by a single-particle quantum walk which is used to entangle two spatially separated, not necessarily correlated, systems. This scheme can be used to entangle any two systems which can interact with the spatial modes entangled during the quantum walk evolution. A notable feature is that we can control the quantum walk dynamics and its ability to localize leads to a substantial control and improvement in the entanglement output.

Introduction – Entanglement is a very useful resource to perform various quantum tasks, a recent review by Horodecki *et al.* [1] and references therein discuss many of these aspects. Several schemes to generate and distribute entanglement between different systems have also been proposed (see [2] for some of the earlier schemes). In most of the schemes, two systems are first entangled and spatially propagated away from each other. Such spatially separated states can be used for quantum communication protocols, for example, quantum cryptography [3] and quantum teleportation [4]. During the process of spatial separation of the entangled system, the degree of entanglement gets degraded. Generation of entanglement when the systems are spatially separated would be one of the methods to circumvent the depletion of entanglement.

In this letter, we present a new scheme to efficiently generate entanglement between two spatially separated systems. It has been shown by some of the present authors that a quantum walk evolution of a particle in a one-dimensional lattice, after sufficient number of walk steps, results in the entanglement of the lattice sites [5]. We make use of this spatial entanglement which by itself does not have much physical significance to entangle two spatially separated, initially separable systems. Control over the quantum walk dynamics, by controlling the quantum coin operation has been shown in Ref. [6]. Its ability to localize the quantum walk at different lattice sites simultaneously, which has been discussed in this letter for the first time, leads to a substantial control and improvement in the entanglement output. This is a generic scheme which can be implemented to entangle any two systems that can interact in a specific way with the spatial modes entangled by the quantum walk. Experimental implementation of quantum walk has been reported with samples in nuclear magnetic resonance (NMR) system [7, 8], in the continuous tunneling of light fields through waveguide lattices [9], in the phase space of trapped ions [10, 11], with single neutral atoms in optically trapped atoms [12], and with single photons [13]. There are various other schemes proposed to implement quantum walk in other systems [14]. Using our

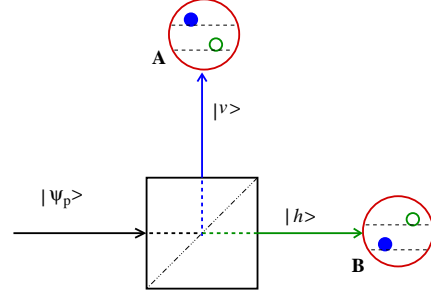


FIG. 1: Photon in state $|\Psi_p\rangle$ when passed through the beam splitter gets separated making h and v spatial modes which are entangled. Two two-level atoms (A and B) in ground state can be entangled by interacting them with the two modes separately.

proposal, all these systems have the potential to generate entanglement between two spatially separated, uncorrelated systems.

Toy model – Before proceeding with the description of our scheme, we will present the basic idea using a simple model involving a beam splitter, a photon and two two-level atoms as systems A and B. The aim is to generate entanglement between A and B. First, a photon in the initial state $|\Psi_p\rangle$ is passed through the beam splitter. The state of the photon passing through the beam splitter are spatially separated into the horizontal (h) and vertical (v) spatial modes and it can be written as $|\psi_s\rangle = \alpha|h\rangle + \beta|v\rangle$, such that $|\alpha|^2 + |\beta|^2 = 1$, where $|\alpha|^2$ and $|\beta|^2$ represent the probability of finding the photon in the horizontal and vertical mode, respectively. For convenience, we can rewrite the state of the spatial modes in terms of the number of photons, that is,

$$|\psi_s\rangle = \alpha|10\rangle_{hv} + \beta|01\rangle_{hv}. \quad (1)$$

The state $|10\rangle_{hv}$ ($|01\rangle_{hv}$) represents one photon in the horizontal (vertical) mode and no photon in vertical (horizontal) mode. This state is entangled unless α or β is zero. This entanglement in the spatial mode can be used to entangle A and B. This is done by placing atoms A

and B initially in the ground state ($|g\rangle_A$ and $|g\rangle_B$) at the two exit points of the photon coming from the beam splitter. The conditions are such that if the photon is in a vertical mode the atom A will get excited ($|e\rangle_A$) and if the photon is in a horizontal mode the atom B will get excited ($|e\rangle_B$). The final collective state of these two atoms can be written as:

$$|\psi_a\rangle = \alpha|eg\rangle_{AB} + \beta|ge\rangle_{AB}. \quad (2)$$

This provides a very simple model of generating entanglement between two atomic systems from the entanglement between the spatial modes. Its pictorial representation is as in Fig. 1. The importance of this scheme is reflected from the fact that a photon in an entangled state of two spatial modes is used to generate entanglement between two spatially separated atoms. Entanglement between two spatial modes by itself is not much of use to perform quantum tasks but using this, the entanglement generated between two systems can be used to perform useful quantum information tasks.

The new scheme we propose below has the ingredients of this toy model namely, generation of entanglement between the spatial modes and using it to entangle two spatially separated physical systems. First, we will discuss the spatial entanglement obtained due to the interference effect during the evolution of the quantum walk. To show the analytics for spatial entanglement we consider two fixed lattice sites after first three steps of the walk. The interactions used to generate entanglement between two uncorrelated systems A and B from the spatially entangled modes of the quantum walk are then discussed. To entangle distantly located A and B , we need to maximize the spatial entanglement between the two distant lattice sites. We discuss the protocol to maximize the spatial entanglement between two distant lattice sites by controlling the quantum walk to localize around those lattice points with a good degree of interference effect. Once the walk is localized, interactions are again used to generate entanglement between A and B , spatially separated, and is shown numerically.

Spatial entanglement using single-particle quantum walk – Discrete-time quantum walk is defined on a *coin* Hilbert space \mathcal{H}_c and *position* Hilbert space \mathcal{H}_p . In one dimension, \mathcal{H}_c is spanned by the basis states $|0\rangle$ and $|1\rangle$ and \mathcal{H}_p is spanned by the basis state $|\psi_j\rangle$, $j \in \mathbb{Z}$. Each step of quantum walk on a particle initially in superposition of the coin state at origin ($j = 0$) given by,

$$|\Psi_{in}\rangle = \frac{1}{\sqrt{2}}[|0\rangle + i|1\rangle] \otimes |\psi_0\rangle \quad (3)$$

is implemented by applying a conditional shift operator S followed by the quantum coin operation C . The shift operator S can be defined such that the state $|0\rangle$ moves

to the left and $|1\rangle$ moves to the right,

$$S = |0\rangle\langle 0| \otimes \sum_{j \in \mathbb{Z}} |\psi_{j-1}\rangle\langle \psi_j| + |1\rangle\langle 1| \otimes \sum_{j \in \mathbb{Z}} |\psi_{j+1}\rangle\langle \psi_j|. \quad (4)$$

Quantum coin operation C is applied after the shift operator S to evolve the state at the new position to the superposition of the coin basis states [15–17]. The quantum coin operation can be any element from the $U(2)$ group and it has been shown in Ref.[6] that additional control over the dynamics of the quantum walk can be achieved by choosing a three parameter $U(2)$ coin operation

$$C_{\xi, \theta, \zeta} \equiv \begin{pmatrix} e^{i\xi} \cos(\theta) & e^{i\zeta} \sin(\theta) \\ e^{-i\zeta} \sin(\theta) & -e^{-i\xi} \cos(\theta) \end{pmatrix}. \quad (5)$$

For simplicity we choose Hadamard operator $H = C_{0, \frac{\pi}{2}, 0} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ as the quantum coin operation to show the spatial entanglement after iterating the quantum walk for three steps. After the first three steps of quantum walk, the state of the particle will be:

$$|\Psi_3\rangle = S(H \otimes \mathbb{1})S(H \otimes \mathbb{1})S|\Psi_{in}\rangle = \frac{1}{2\sqrt{2}}[|0\rangle|\psi_{-3}\rangle + \sqrt{3}|\chi_{-1}\rangle|\psi_{-1}\rangle + \sqrt{3}|\chi_1\rangle|\psi_1\rangle + i|1\rangle|\psi_3\rangle], \quad (6)$$

where $|\chi_{-1}\rangle = [(1+i)|0\rangle + |1\rangle]/\sqrt{3}$ and $|\chi_1\rangle = [-i|0\rangle - (1-i)|1\rangle]/\sqrt{3}$. In order to obtain the state of the lattice we can redefine the occupied position states in the following way: the position state $|\psi_{-3}\rangle$ as $|1000\rangle$, $|\psi_{+3}\rangle$ as $|0001\rangle$. Lattice state can then be written as:

$$|\Psi_3\rangle_{latt} = \frac{1}{2\sqrt{2}}[|0\rangle|1000\rangle + \sqrt{3}|\chi_{-1}\rangle|0100\rangle + \sqrt{3}|\chi_1\rangle|0010\rangle + i|1\rangle|0001\rangle]. \quad (7)$$

Let us concentrate on the lattice sites -1 and $+1$ only. The reduced density matrix of these two lattice sites after tracing out the other sites and the coin degree of freedom will be:

$$\rho_{-1,1} = \frac{1}{8}[2|00\rangle\langle 00| + 3|10\rangle\langle 10| + 3|01\rangle\langle 01| + 3\gamma|10\rangle\langle 01| + 3\gamma^*|01\rangle\langle 10|] = \frac{1}{8} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 3\gamma & 0 \\ 0 & 3\gamma^* & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (8)$$

where $\gamma = \langle \chi_1 | \chi_{-1} \rangle$. The partial transpose of the above matrix will always be non-positive. Therefore, the reduced density matrix $\rho_{-1,1}$ represents an entangled state. This shows that the lattice sites -1 and $+1$ are entangled. The next task is to use this spatially entangled states to generate entanglement systems A and B .

Generation of entanglement between two spatially separated systems from spatially entangled modes – To achieve

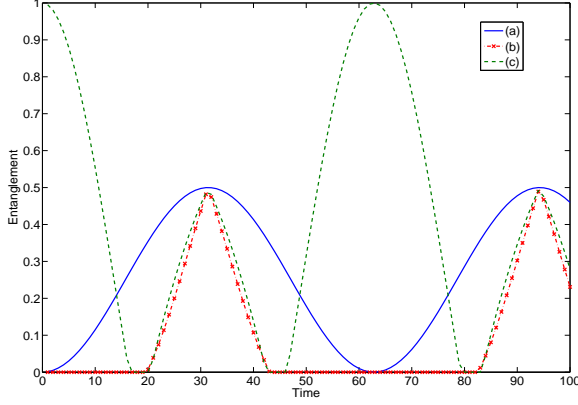


FIG. 2: The evolution of generated entanglement between A and B with different initial states under the influence of interaction with entangled lattice sites ± 1 . (a) $\rho_{AB} = |00\rangle$, (b) $\rho_{AB} = |11\rangle$ and (c) $\rho_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

the purpose of generating entanglement between A and B , they need to interact with the two different lattice sites on which the quantum walk is being performed. The interaction of systems A and B with lattice depends upon the nature of the system on which quantum walk is being performed and the properties of the system A and B . For instance, if we consider an optical system [13] in which a single photon is performing quantum walk and the systems A and B are two-level atoms capable of absorbing the photon and get excited. The interaction Hamiltonian for this model can be written as: $H_I = g(\sigma^+ a + \sigma^- a^\dagger)$, where σ^\pm are the spin raising and lowering operators and a, a^\dagger are the creation and annihilation operators, respectively. Another example could be a one-dimensional lattice consisting of spin- $\frac{1}{2}$ particles. In such systems the spin hops from one lattice site to another in a quantum walk evolution. By using Jordan-Wigner transformation [18] a spin- $\frac{1}{2}$ system can be mapped to a spinless fermionic system. Therefore, a one-dimensional spin- $\frac{1}{2}$ lattice with all spins but one, pointing downward can be viewed as a system consisting of a single spinless fermion. By attaching an extra coin degree of freedom with the fermion one can perform quantum walk. Now if systems A and B are also spin- $\frac{1}{2}$ particles interacting with spins at $\pm l$ lattice sites, the interaction Hamiltonian can be written as $H_I = \hbar\omega(\sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+)$. There can be other kinds of interactions as well but for the present demonstration we will consider the spin case only. The time evolution operator for this setting can be written as: $U_I \exp(-i\omega t(\sigma_- \otimes \sigma_+ + \sigma_+ \otimes \sigma_-))$.

After letting A interact with spin at $-l$ and B with spin at $+l$ for time t , the state of the system AB can be

written as:

$$\rho_{AB}(t) = \sum_{ij} A_{ij} \rho_{AB} A_{ij}^\dagger. \quad (9)$$

Here $A_{ij} = \sqrt{\lambda_j} \langle i|W|\eta_j\rangle$ are Kraus operators and: $\{|i\rangle\}$ is an orthonormal basis in $\mathcal{H}_{-l} \otimes \mathcal{H}_{+l}$, $\{\lambda_j, |\eta_j\rangle\}_{j=1}^4$ forms the set of eigenvectors and eigenvalues for $\rho_{-l,+l}$ and ρ_{AB} is the initial state of the system AB . The unitary operator $W = P[U_I \otimes U_I]P^T$ where P is a permutation operator such that $P[\mathcal{H}_{-l} \otimes \mathcal{H}_A \otimes \mathcal{H}_{+l} \otimes \mathcal{H}_B] = \mathcal{H}_{-l} \otimes \mathcal{H}_{+l} \otimes \mathcal{H}_A \otimes \mathcal{H}_B$.

In Fig. 2 we show the evolution of entanglement, using concurrence [19] as entanglement measure, between systems A and B interacting with the lattice sites -1 and $+1$ after three step of quantum walk for different initial state. We observe that the amount of entanglement depends on the initial state of the system A and B and separable states achieve their maximum entanglement at the same time.

Scaling and localization at different lattice sites – Degree of spatial entanglement between the two lattice sites depends largely on the degree of interference of the coin state during the quantum walk evolution and its amplitude at that particular lattice site [5]. The quantum walk evolution with good degree of interference results in spread of the amplitude in position space. Choosing value of θ very close to zero in the coin operation given by Eq. (5), minimizes the spread of amplitude but evolves with a very low interference effect. Therefore, both the above cases result in less spatial entanglement. To maximize the amount of spatial entanglement one needs to control the quantum walk evolution in such a way that its amplitude is localized, with a good degree of interference.

To further understand the contribution of localization of quantum walk to increase the degree of spatial entanglement, we will consider the lattice state after t steps of quantum walk. The state can be written as $|\Psi_t\rangle_{latt} = \sum_j |\phi_j\rangle \otimes |j\rangle$, where $|j\rangle$ represents a state where lattice site j is occupied with other sites being empty and $|\phi_j\rangle$ represent the corresponding coin state. Let us say we are interested in particular lattice sites $\pm l$. Then the states other than $|\pm l\rangle$ will contribute to $|00\rangle\langle 00|$ in the reduced density matrix $\rho_{\pm l}$ which is of the form given by Eq. (8) with off diagonal terms $\langle \phi_l | \phi_{-l} \rangle$ and diagonal terms depending on the amplitude of all $|j\rangle$ states. If the amplitude of all the $|j\rangle$ states except $|\pm l\rangle$ is zero, then the coefficient of $|00\rangle\langle 00|$ will be zero and the amount of entanglement will depend on $\langle \phi_l | \phi_{-l} \rangle$. Therefore, in localized states, that is, the states in which the amplitude is localized in a very narrow lattice space, the coefficient of $|00\rangle\langle 00|$ will be very small and hence the amount of spatial entanglement will be more.

Another way to look at this would be to consider the rate of entanglement generation between lattice points, say $\pm l$, at which localization has been achieved. Localization of the walk enables us to think of an equivalent

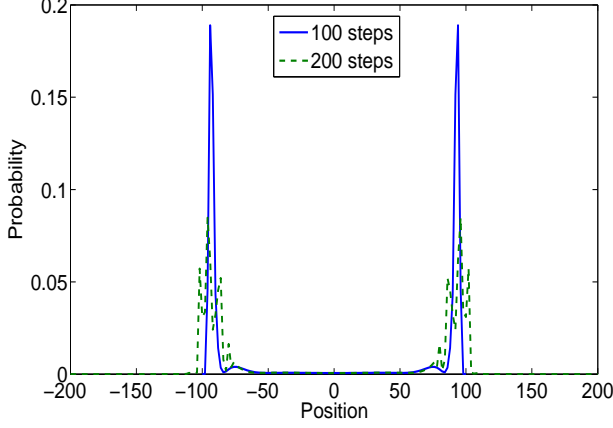


FIG. 3: Localization of quantum walk around two lattice sites, -95 and $+95$ by choosing a specific combination of coin operations during the evolution. Localization contributes to the increase in the degree of spatial entanglement which can be used to entangle two spatially separated systems A and B .

situation of these amplitudes as say, a block M of spins and the rest of the lattice as a block of spins N , with the evolution being generated by the local Hamiltonian. Then the entanglement generated in block M , in a finite time, scales like the perimeter of the block [20]. Without localization, to consider entanglement between the same two points from this point of view, one would have to consider a number of blocks, depending upon the number of sites having finite amplitude, which due to the bound on entropy generation as the block perimeter results in a decrease in entanglement between M and N . This compliments our above picture.

Localization of quantum walk at the origin has been discussed in Refs. [21, 22]. Here, we briefly discuss a way to localize the quantum walk at particular lattice sites. In Ref. [22], quantum walk using randomly picked coin operation for each step of walk with $\theta \in \{\pi/4, \pi/2\}$ for $C_{0,\theta,0} = C_\theta$ was shown to localize the quantum walk at the initial position. We also know that choosing θ to be very small, $0 < \theta < \pi/36 = 5^\circ$ will result in two peaks of amplitudes moving away from each other with minimal interference [6]. The combination of the above two, coin operation with θ_1 being very small, say $\pi/36$ and coin operation with randomly picked coins for each step from set $\theta_2 \in \{\pi/4, \pi/2\}$ can be used to localize the walk at particular lattice sites. In general we can write the t -step operation of quantum walk as :

$$S(C_{\theta_t} \otimes \mathbb{1}) \cdots S(C_{\theta_1} \otimes \mathbb{1}) \cdots S(C_{\theta_0} \otimes \mathbb{1}) |\Psi_{in}\rangle. \quad (10)$$

To localize the amplitude during quantum walk at positions $+l$ and $-l$, we need to choose the coin operation θ_1 for up to l number of iteration followed by θ_2 for any number of operations after that. Fig. 3 shows the localization of the amplitude at positions -95 and $+95$.

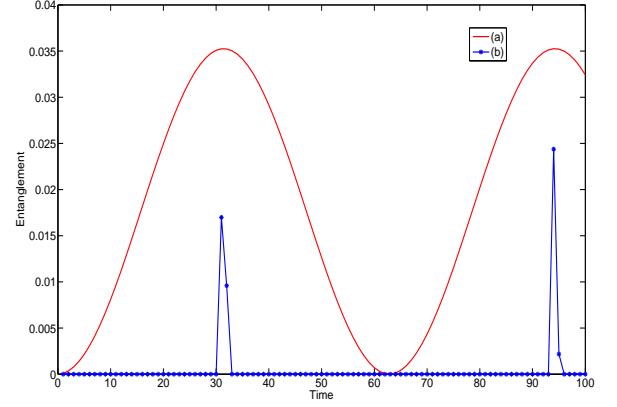


FIG. 4: The evolution of generated entanglement between A and B with different initial states under the influence of interaction with entangled lattice sites ± 95 . Entanglement was generated after the walk was localized at site ± 95 . (a) $\rho_{AB} = |00\rangle$ and (b) $\rho_{AB} = |11\rangle$.

First 95 steps of quantum walk was evolved using θ_1 and later on randomly picked coins for each step from $\theta_2 \in \{\pi/4, \pi/2\}$ was used to localize the walk around that lattice point. We note that the peaks do not move away even after 200 steps of walk. One should avoid using $\theta_1 = 0$, though it gives very sharp peaks at position $+l$ and $-l$ after l steps. That is because, the walk would have evolved without interfering and hence spatial entanglement generated from ballistic evolution will be zero. Other degrees of freedom ξ and ζ in the coin operation $C_{\xi,\theta,\zeta}$ (Eq. 5) can be used to introduce asymmetry in the localized peaks at position $+l$ and $-l$. In Fig. 4 we show the evolution of entanglement with time between systems A and B interacting with the lattice sites -95 and $+95$ (localized) after 200 steps of the quantum walk. Although the amount of entanglement generated in this case is less than that of the amount generated in Fig. 2, qualitatively both evolutions are identical.

Conclusion – We have presented a scheme to generate entanglement between two distantly located systems A and B using entangled spatial modes due to quantum walk. Amount of entanglement generated depends directly on the spatial entanglement and the initial state of A and B . A scheme to localize quantum walk at desired lattice sites has been demonstrated using randomized quantum coin operations. This localized quantum walk has been used to increase the distance and entanglement between A and B . This scheme can be used to entangle any two or more particles in a physical system by interacting them with an external quantum walk system. The work on implementing this scheme on large physical system is in progress.

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